TRIGONOMETRIC INTEGRALS

The starting point of this section is to see that:

$$\int \sin^n x \cos x \, dx = \frac{\sin^{n+1} x}{n+1} + C \qquad \int \tan^n x \sec^2 x \, dx = \frac{\tan^{n+1} x}{n+1} + C$$

$$\int \cos^n x \sin x \, dx = -\frac{\cos^{n+1} x}{n+1} + C \qquad \int \sec^n x \sec x \tan x \, dx = \frac{\sec^{n+1} x}{n+1} + C$$

The above formulas hold for $n \neq -1$

1. Integrals involving powers of sine and cosine

When the integrand involves powers of sine and cosine, we want to convert it to a form where we have only one sine factor (and the rest in terms of cosine) and vice versa, so we can apply the formulas from above.

The identity

$$\boxed{\cos^2 x + \sin^2 x = 1} \tag{1}$$

allows us to change between even powers of sine and cosine.

Exercise 1. Express $\cos^6 x$ in terms of $\sin x$.

For example if we were asked to evaluate

$$\int \sin^3 x \, dx$$

We would write $\sin^3 x$ as $\sin x \cdot \sin^2 x$ and use the identity (1) to write it as $\sin x \cdot (1 - \cos^2 x)$. Then use the substitution $u = \cos x$ to compute the integral.

Exercise 2. Evaluate

$$\int \sin^3 x \, dx$$

Now we want to use the same idea to evaluate integrals of the form

$$\int \sin^m x \cos^n x \, dx$$

- If the power of $\sin x$ is odd, save one $\sin x$ factor and use $\sin^2 x = 1 - \cos^2 x$, to express the remaining powers of sine in terms of cosine. Then use the substitution $u = \cos x$.

Exercise 3. Evaluate

$$\int \sin^2 x \cos^3 x \, dx$$

- If the power of $\cos x$ is odd, save one $\cos x$ factor and use $\cos^2 x = 1 - \sin^2 x$, to express the remaining powers of $\cos x$ in terms of $\sin x$. Then use the substitution $u = \sin x$.

Exercise 4.

$$\int \sin^5 x \cos^4 x \, dx$$

- If the powers of $\sin x$ and $\cos x$ are both even, then use will need to use the double-angle (half-angle) identities:

$$\boxed{\cos^2 x = \frac{1}{2}(1 + \cos 2x)} \qquad \boxed{\sin^2 x = \frac{1}{2}(1 - \cos 2x)} \qquad \boxed{\sin(2x) = 2 \sin x \cos x}$$

The following formulas will very useful when computing integrals

$$\boxed{\int \cos(ax) \, dx = \frac{1}{a} \sin(ax) + C} ; \qquad \boxed{\int \sin(ax) \, dx = -\frac{1}{a} \cos(ax) + C} \quad a \neq 0$$

Exercise 5. Evaluate

$$\int \cos^4 x \, dx$$

Exercise 6. Evaluate
$$\int \cos^2 x \sin^2 x \, dx$$

2. Integration involving powers of $\sec x$ and $\tan x$

We are interested in evaluating integrals of the form
$$\int \sec^m x \tan^n x \, dx$$

The strategy is similar to the one discussed above, and the identity that allows us to change between even powers of $\sec x$ and $\tan x$ is
$$\boxed{\sec^2 x - \tan^2 x = 1}$$

- If the power of $\sec x$ is even, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$. Then use the substitution $u = \tan x$.

Exercise 7. Evaluate
$$\int \sec^4 x \tan^4 x \, dx$$

- If the power of $\tan x$ is odd, save a factors of $\sec x \tan x$ and use the identity $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$. Then use the substitution $u = \sec x$.

Exercise 8. Evaluate

$$\int \tan^5 \theta \sec^7 \theta \, d\theta$$

- Otherwise there is no direct method. We can use trigonometric identities or integration by parts. We do however have the following formulas:

$$\boxed{\int \tan x \, dx = \ln|\sec x| + C} \qquad \boxed{\int \sec x \, dx = \ln|\sec x + \tan x| + C}$$

Exercise 9. Evaluate

$$\int \tan^3 \theta \, d\theta$$

Exercise 10. Evaluate

$$\int \sec^3 t \; dt$$

1) $\cos^6 x = [\cos^2 x]^3 = (1-\sin^2 x)^3$

$= 1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x$

2) $\int \sin^3 x \, dx = \int \sin^2 x \cdot \sin x dx \longrightarrow$ Since power of sinx is odd save 1 copy of sinx & change the rest to cosx. Use u = cosx

$= \int (1-\cos^2 x) \sin x \, dx$

$u = \cos x \qquad du = -\sin x \, dx \qquad \Rightarrow dx = \frac{du}{-\sin x}$

$= \int (1-u^2) \cancel{\sin x} \frac{du}{-\cancel{\sin x}}$

$= \int u^2 - 1 \, du$

$= \frac{u^3}{3} - u + C$

$= \frac{\sin^3 x}{3} - \sin x + C$

3) $\int \sin^2 x \cos^3 x \, dx$ $\longrightarrow$ Power of cosx is odd

Save a power of cosx, and change evth else to sinx & use $u = \sin x$

$= \int \sin^2 x \cdot \cos^2 x \cos x \, dx$

$= \int \sin^2 x (1-\sin^2 x) \cos x \, dx$

$u = \sin x$, $du = \cos x \, dx$

$= \int u^2 (1-u^2) \, du$

$= \int u^2 - u^4 \, du = \frac{u^3}{3} - \frac{u^5}{5} + C$

$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

4) $\int \sin^5 x \cos^4 x \, dx$ $\rightarrow$ Power of sinx is odd

$= \int \sin^4 x \cos^4 x \sin x \, dx$

$$= \int (\sin^2 x)^2 \cos^4 x \ \sin x \, dx$$

$$= \int (1-\cos^2 x)^2 \cos^4 x \ \sin x \, dx$$

let $u = \cos x$

$du = -\sin x \, dx$

$dx = \dfrac{du}{-\sin x}$

$$= \int (1-u^2)^2 u^4 \sin x \frac{du}{-\sin x}$$

$$= -\int (1-2u^2+u^4)u^4 \, du$$

$$= -\int u^4 - 2u^6 + u^8 \, du = -\left[\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right] + C$$

$$= -\left[\frac{\cos^5 x}{5} - \frac{2\cos^7 x}{7} + \frac{\cos^9 x}{9}\right] + C$$

5) $\int \cos^4 x \, dx \longrightarrow$ Power of $\cos x$ is even, and no $\sin x$ means we need to use the half angle formula $\cos^2 x = \frac{1}{2}\left[1+\cos 2x\right]$

$$= \int \left(\frac{1}{2}\left[1+\cos 2x\right]\right)^2 dx$$

$$= \frac{1}{4}\int 1 + 2\cos 2x + \cos^2(2x) \, dx = \frac{1}{4}\left[\int 1 \, dx + \int 2\cos 2x \, dx + \int \cos^2(2x) \, dx\right]$$

$$= \frac{1}{4}x + \frac{1}{4}\cdot\frac{2\sin 2x}{2} + \frac{1}{4}\underbrace{\int \cos^2(2x) \, dx}$$

Again we have an even power of $\cos(2x)$ means we need to use the formula $\cos^2(2x) = \frac{1}{2}\left(1+\cos(4x)\right)$

$$= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{4}\int \frac{1}{2}(1+\cos 4x) \, dx$$

$$= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}\int 1 + \cos 4x \, dx$$

$$= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{8}\frac{\sin 4x}{4} + C$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{\sin 4x}{32} + C$$

6) $\int \cos^2 x \sin^2 x \, dx \rightarrow$ Even powers of $\cos x, \sin x$

$$= \int \frac{1}{2}\left[1+\cos 2x\right] \cdot \frac{1}{2}\left[1-\cos 2x\right] dx$$

$$= \frac{1}{4}\int 1-\cos^2 2x \, dx$$

$$= \frac{1}{4}\int 1 \, dx \quad - \frac{1}{4}\int \cos^2 2x \, dx$$

$$= \frac{1}{4}x - \frac{1}{4}\int \frac{1}{2}\left[1+\cos 4x\right] dx$$

$$= \frac{1}{4}x - \frac{1}{8}\int 1+\cos 4x \, dx$$

$$= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{8}\,\frac{\sin 4x}{4} + C$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

7) $\int \sec^4 x \tan^4 x \, dx$ $\rightarrow$ Power of $\sec x$ is even so save $\sec^2 x$, change evth in terms of $\tan x$ and use $u = \tan x$

$$= \int \sec^2 x \cdot \sec^2 x \, \tan^4 x \, dx$$

$$= \int \sec^2 x \cdot (1+\tan^2 x) \tan^4 x \, dx$$

$u = \tan x$; $du = \sec^2 x \, dx$ $\Rightarrow dx = \dfrac{du}{\sec^2 x}$

$$= \int \cancel{\sec^2 x} \, (1+u^2) u^4 \frac{du}{\cancel{\sec^2 x}}$$

$$= \int u^4 + u^6 \, du$$

$$= \frac{u^5}{5} + \frac{u^7}{7} + C$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$$

8) $\int \tan^5 \theta \sec^7 \theta \, d\theta$ $\rightarrow$ Power of $\tan\theta$ is odd and we have power of $\sec\theta$, so save a copy of $\sec\theta \tan\theta$

$$= \int \sec\theta \tan\theta \, \tan^4\theta \sec^6\theta \, d\theta$$

$$= \int \sec\theta \tan\theta \, (\tan^2\theta)^2 \sec^6\theta \, d\theta$$

$$= \int \sec\theta \tan\theta \, (\sec^2\theta - 1)^2 \sec^6\theta \, d\theta$$

$u = \sec\theta$

$du = \sec\theta \tan\theta \, d\theta$

$d\theta = \dfrac{du}{\sec\theta \tan\theta}$

$$= \int \sec\theta \tan\theta \, (u^2 - 1)^2 u^6 \frac{du}{\sec\theta \tan\theta}$$

$$= \int (u^4 - 2u^2 + 1) u^6 \, du$$

$$= \int u^{10} - 2u^8 + u^6 \, du$$

$$= \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C$$

$$= \frac{\sec^{11}\theta}{11} - \frac{2\sec^9\theta}{9} + \frac{\sec^7\theta}{7} + C$$

9) $\int \tan^3\theta \, d\theta$

$= \int \tan\theta \cdot \tan^2\theta \, d\theta$

$= \int \tan\theta \, (\sec^2\theta - 1) \, d\theta$

$= \int \tan\theta \sec^2\theta - \tan\theta \, d\theta$

$= \int \tan\theta \sec^2\theta \, d\theta - \int \tan\theta \, d\theta$

~~let~~ let $u = \tan\theta$

$du = \sec^2\theta \, d\theta$

$= \int u \, du - \ln|\sec\theta|$

$= \frac{u^2}{2} - \ln|\sec\theta| + C$

$= \frac{\tan^2\theta}{2} - \ln|\sec\theta| + C$

10) $\int \sec^3 t \, dt$

$= \int \sec t \cdot \sec^2 t \, dt$ → Integration by parts

$u = \sec t \quad dv = \sec^2 t \, dt$

$du = \sec t \tan t \, dt \; ; \; v = \tan t$

$$\int \sec^3 t \, dt = \sec t \tan t - \int \tan t \sec t \tan t \, dt$$

$$= \sec t \tan t - \int \tan^2 t \sec t \, dt$$

$$= \sec t \tan t - \int (\sec^2 t - 1) \sec t \, dt$$

$$\int \sec^3 t \, dt = \sec t \tan t - \int \sec^3 t \, dt + \int \sec t \, dt$$

Call $I = \int \sec^3 t \, dt$

$$I = \sec t \tan t - I + \ln|\sec t + \tan t| + C$$

$$2I = \sec t \tan t + \ln|\sec t + \tan t| + C \Rightarrow I = \frac{1}{2}\left[\sec t \tan t + \ln|\sec t + \tan t|\right] + C$$